arXiv:1104.2054v1 [math.DS] 11 Apr 2011

DYNAMICS OF NON ABELIAN AFFINE HOMOTHETIES GROUP OF $\mathbb{C}^n$

YAHYA N'DAO AND ADLENE AYADI

Abstract. In this paper, we study the action of non abelian group G generated by affine homotheties on $\mathbb{C}^n$. We prove that there exist a subgroup Λ_G of $\mathbb{C}^*$, a G-invariant affine subspace E_G of $\mathbb{C}^n$ and $a \in E_G$ such that $\overline{G(z)} = \overline{\Lambda_G}(z-a) + E_G$ for every $z \in \mathbb{C}^n$. In particular, $\overline{G(z)} = E_G$ for every $z \in E_G$ and if $E_G \neq \mathbb{C}^n$, every orbit in $U = \mathbb{C}^n \backslash E_G$ is minimal in U. Moreover, we characterize the existence of dense orbit of G. As a consequence of the case $n = 1$, we describe the action of affine rotations groups of $\mathbb{R}^2$.

1. Introduction

A map $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is called an affine homothety if there exists $\lambda \in \mathbb{C}\backslash\{0,1\}$ and $a \in \mathbb{C}^n$ such that $f(z) = \lambda(z-a)+a$ for every $z \in \mathbb{C}^n$. (i.e. $f = T_a \circ (\lambda.id_{\mathbb{C}^n}) \circ T_{-a}$, $T_a : z \longmapsto z + a$, $id_{\mathbb{C}^n}$ the identity map of $\mathbb{C}^n$). Write $f = (a, \lambda)$ and we call a *the center* of f and λ the ratio of f.

Denote by:

• $\mathcal{H}(n, \mathbb{K})$ the group generated by all affine homotheties of $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). i.e.

$$\mathcal{H}(n, \mathbb{K}) := \{\ f : z \ \longmapsto\ \lambda z + a;\ a \in \mathbb{K}^n,\ \lambda \in \mathbb{K}^*\}.$$

• $\mathcal{R}_n$ the subgroup of $\mathcal{H}(n, \mathbb{C})$ generated by all affine rotations of $\mathbb{C}^n$. i.e.

$$\mathcal{R}_n := \left\{\ f : z \ \longmapsto\ e^{i\theta} z + a;\ a \in \mathbb{C}^n,\ \theta \in \mathbb{R}\right\}.$$

• $\mathcal{T}_n$ the subgroup of $\mathcal{H}(n, \mathbb{C})$ generated by all translations of $\mathbb{C}^n$.

• Let $H_2 = \left(\frac{\pi}{2} + \pi\mathbb{Z}\right) \cup (\pi\mathbb{Z})$, $F_2 = \{e^{ix},\ \ x \in H_2\}$ and

$$\mathcal{S}_2\mathcal{R}_n := \left\{\ f = (a, e^{i\theta}) \in \mathcal{R}_n;\ \theta \in H_2, a \in \mathbb{C}^n\right\}.$$

• Let $H_3 = \left(\frac{\pi}{3} + \pi\mathbb{Z}\right) \cup \left(-\frac{\pi}{3} + \pi\mathbb{Z}\right) \cup (\pi\mathbb{Z})$, $F_3 = \{e^{ix},\ \ x \in H_3\}$ and

$$\mathcal{S}_3\mathcal{R}_n := \left\{\ f = (a, e^{i\theta}) \in \mathcal{R}_n;\ \theta \in H_3, a \in \mathbb{C}^n\right\}.$$

We have F_2 and F_3 are finite, $\mathcal{S}_2\mathcal{R}_n$ and $\mathcal{S}_3\mathcal{R}_n$ are subgroups of $\mathcal{H}(n, \mathbb{C})$ containing $\mathcal{T}_n$.

• $\mathcal{S}\mathcal{R}_n := \mathcal{S}_2\mathcal{R}_n \cup \mathcal{S}_3\mathcal{R}_n$.

2000 *Mathematics Subject Classification.* 37C85.

Key words and phrases. Homothety, Rotation, orbit, density, minimal, non abelian, action, dynamic,...

This work is supported by the research unit: systèmes dynamiques et combinatoire: 99UR15-15.

We say a *group of affine homotheties* of $\mathbb{C}^n$ any subgroup of $\mathcal{H}(n,\mathbb{C})$.

Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$. There is a natural action $\mathcal{H}(n,\mathbb{C})\times\mathbb{C}^n : \longrightarrow \mathbb{C}^n$. $(f,v) \longmapsto f(v)$. For a vector $v\in\mathbb{C}^n$, denote by $G(v) := \{f(v) : f\in G\}\subset\mathbb{C}^n$ the *orbit* of G through v. A subset $A\subset\mathbb{C}^n$ is called *G-invariant* if $f(A)\subset A$ for any $f\in G$; that is A is a union of orbits and denote by $\overline{A}$ (resp. $\overset{\circ}{A}$) the closure (resp. interior) of A.
If U is an open G-invariant set, the orbit $G(v)\subset U$ is called *minimal in U* if $\overline{G(v)}\cap U = \overline{G(w)}\cap U$ for every $w\in\overline{G(v)}\cap U$.

We say that H is an *affine subspace* of $\mathbb{C}^n$ with dimension p if $H = E + a$, for some $a\in\mathbb{C}^n$ and some vector subspace E of $\mathbb{C}^n$ with dimension p. For every subset A of $\mathbb{C}^n$, denote by $vect(A)$ (resp. $Aff(A)$) the vector (resp. affine) subspace of $\mathbb{C}^n$ generated by all elements of A.

Denote by:
- $\Lambda_G := \{\lambda :\ f = (a,\lambda)\in G\}$. It is obvious that Λ_G is a subgroup of $\mathbb{C}^*$ (see Lemma 2.4).
- $\mathrm{Fix}(f) := \{z\in\mathbb{C}^n :\ f(z) = z\}$, for every $f\in\mathcal{H}(n,\mathbb{C})$. See that $Fix(f) = \emptyset$ if $f\in\mathcal{T}_n$ and $Fix(f) = a$ if $f = (a,\lambda)\in\mathcal{H}(n,\mathbb{C})\backslash\mathcal{T}_n$.
- $\Gamma_G := \bigcup\limits_{f\in G\backslash\mathcal{T}_n}\mathrm{Fix}(f)$. Since G is non abelian then $G\backslash\mathcal{T}_n\neq\emptyset$, so $\Gamma_G\neq\emptyset$.
- $G_1 := G\cap\mathcal{T}_n$ is a subgroup of $\mathcal{T}_n$.
- $G_1(0) = \{f(0),\ f\in G_1\}$.
- $E_G = Aff(\Gamma_G\cup G_1(0))$ the affine subspace of $\mathbb{C}^n$ generated by $\Gamma_G\cup G_1(0)$.
- $S^1 = \{z\in\mathbb{C} :\ \ |z| = 1\}$.

Remark that $E_G\neq\emptyset$ since $\Gamma_G\neq\emptyset$ and $G_1(0)\neq\emptyset$.

In [5], we have described the action of non abelian subgroup of $\mathcal{H}(n,\mathbb{R})$. This paper can be viewed as continuation of that work.
I learned that Zhukova have proved in [1] similar results to Lemma 3.1, Proposition 4.2 and Corollary 1.2.(ii), in the real case. The methods of proof in [1] and in this paper are quite different and have different consequences.

In [2], Arek Goetz investigates noninvertible piecewise isometries in $\mathbb{R}^2$ with the particular interest on the maximal invariant sets and ω-limit sets. Unlike in [3], the induced isometries T_0 and T_1 of its system $T : X\longrightarrow X$ are not translations but rotations. The partition P consists of two atoms: P_0, the open left halfplane and P_1, the closed right halfplane.

Our principal results can be stated as follows:

Theorem 1.1. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$ such that $\Lambda_G\backslash\mathbb{R}\neq\emptyset$. Then :*
(1) *If $G\backslash\mathcal{SR}_n\neq\emptyset$, one has:*
   (i) *$\overline{G(z)} = E_G$, for every $z\in E_G$.*
   (ii) *if $U = \mathbb{C}^n\backslash E_G\neq\emptyset$, there exists $a\in E_G$ such that $\overline{G(z)} = \overline{\Lambda_G}(z-a) + E_G$, for every $z\in U$.*
(2) *If $G\subset\mathcal{SR}_n$, one has:*
   (i) *$G\subset\mathcal{S}_i\mathcal{R}_n$ for some $i\in\{2,3\}$.*

(ii) $\overline{G(z)} = F_i z + \overline{G(0)}$, *for every* $z \in \mathbb{C}^n$.

Corollary 1.2. *Under notations of Theorem 1.1. One has:*
(1) If $G\backslash\mathcal{SR}_n \neq \emptyset$ *and* $U \neq \emptyset$, *then:*

(i) *Every orbit in* U *is minimal in* U.
(ii) *If* $G\backslash\mathcal{R}_n \neq \emptyset$, *then* E_G *is a minimal set of* G *in* $\mathbb{C}^n$ *contained in the closure of every orbit of* G.
(iii) *All orbits in* U *are homeomorphic.*

(2) *If* $G \subset \mathcal{SR}_n$, *then* G *has a dense orbit if and only if* $\overline{G_1(0)} = \mathbb{C}^n$.

Corollary 1.3. *Let* G *be a non abelian subgroup of* $\mathcal{H}(n, \mathbb{C})$ *with* $\Lambda_G\backslash\mathbb{R} \neq \emptyset$ *and* $G\backslash\mathcal{SR}_n \neq \emptyset$, *then* G *has no discrete orbit.*

Corollary 1.4. *Let* G *be a non abelian subgroup of* $\mathcal{H}(n, \mathbb{C})$ *with* $\Lambda_G\backslash\mathbb{R} \neq \emptyset$ *and* $G\backslash\mathcal{SR}_n \neq \emptyset$. *Then the following assertions are equivalents:*

(1) G *has a dense orbit in* $\mathbb{C}^n$.
(2) *Every orbit of* U *is dense in* $\mathbb{C}^n$.
(3) G *satisfies one of the following:*
  (i) $E_G = \mathbb{C}^n$
  (ii) $dim(E_G) = n - 1$ *and* $\overline{\Lambda_G} = \mathbb{C}$.

Theorem 1.5. *Let* G *be a non abelian group generated by two affine rotations* R_θ *and* $R'_{\theta'}$ *of* $\mathbb{R}^2$, *having angle respectively* θ *and* θ'. *Then:*
(1) *every orbit of* G *is dense in* $\mathbb{R}^2$ *if and only if there is one of the following:*

(i) $(H_2 \cup H_3)\backslash\{\theta, \theta'\} \neq \emptyset$.
(ii) $\theta \in H_i$ *and* $\theta' \in H_j$ *with* $i \neq j$, $i, j \in \{2, 3\}$.

(2) *every orbit of* G *is closed and discrete in* $\mathbb{R}^2$ *if and only if* $G_1(0)$ *is closed and discrete with* $\theta, \theta' \in H_i$ *for some* $i \in \{2, 3\}$.

Remark 1.6. If G is a non abelian subgroup of $\mathcal{H}(n, \mathbb{C})$ with $\Lambda_G \subset \mathbb{R}$, then it can be considered as a subgroup of $\mathcal{H}(2n, \mathbb{R})$, by identifying $\mathbb{C}^n$ to $\mathbb{R}^{2n}$. So Theorem 1.1 has the same form of Theorem1.1 in the real case (see [5]).

Corollary 1.7. *If* G *is a non abelian subgroup of* $\mathcal{H}(n, \mathbb{C})$ *generated by n-2 affine maps, it has no dense orbit.*

This paper is organized as follows: In Section 2, we introduce some preliminaries Lemmas. In Section 3, we characterize the case $n = 1$ and we prove Theorem 1.5. Section 4 is devoted to given some results in the case $G\backslash\mathcal{SR}_n \neq \emptyset$. In Section 5, we characterize any subgroup of $\mathcal{SR}_n$. In Section 6, we prove Theorem 1.1, Corollaries 1.2, 1.3, 1.4 and 1.7. In Section 7, we give three examples.

2. Preliminaries Lemmas

Lemma 2.1.

(i) *Let* $f = (a, \alpha)$, $g = (b, \beta) \in \mathcal{H}(n, \mathbb{C})\backslash\mathcal{T}_n$ *then* $f \circ g = g \circ f$ *if and only if* $a = b$ *or* $\alpha = 1$ *or* $\beta = 1$.

(ii) *If* $Fix(f) = Fix(g)$ *then* $f \circ g = g \circ f$.

(iii) *Let* G *be a non abelian subgroup of* $\mathcal{H}(n, \mathbb{C})$, *then there exist* $f = (a, \alpha)$, $g = (b, \beta) \in G\backslash\mathcal{T}_n$ *such that* $a \neq b$.

Proof. (i) $f \circ g(x) = g \circ f(x)$, for every $x \in \mathbb{R}^n$, if and only if

$$\begin{aligned} & \lambda(\mu(x-b)+b-a)+a = \mu(\lambda(x-a)+a-b)+b, \\ \iff & -\lambda\mu b + \lambda(b-a) + a = -\mu\lambda a + \mu(a-b) + b, \\ \iff & (a-b)(\lambda\mu - \lambda - \mu + 1) = 0 \\ \iff & (a-b)(\lambda-1)(\mu-1) = 0. \end{aligned}$$

So the results follows.

(ii) There are two cases:
• If $Fix(f) = Fix(g) = \emptyset$ then $f = T_a$ and $g = T_b$ for some $a, b \in \mathbb{R}^n$, so $f \circ g = g \circ f$.
• If $Fix(f) = Fix(g) = a$, then $T_a \circ f \circ T_{-a} = \lambda id$ and $T_a \circ g \circ T_{-a} = \mu id$, for some $\lambda, \mu \in \mathbb{C}^*$, so $f \circ g = g \circ f$.
The proof of (iii) results from (ii) since G is non abelian. □

Lemma 2.2. ([5], *Lemma 2.3*) *Let* $\mathcal{B} = (a_1, \dots, a_n)$ *be a basis of* $\mathbb{C}^n$. *Then* $\mathcal{A}ff(\mathcal{B})$ *is defined by*

$$\mathcal{A}ff(\mathcal{B}) := \left\{ z = \sum_{k=1}^{n} \alpha_k a_k : \ \alpha_k \in \mathbb{C}, \ \sum_{k=1}^{n} \alpha_k = 1 \right\}.$$

Remark 2.3. As consequence of Lemma 2.2, if E_G contains $a, a_1, \dots, a_n$ such that $(a_1, \dots, a_n)$ is a basis of $\mathbb{C}^n$ and $a = \sum\limits_{k=1}^{n} \alpha_k a_k$ with $\sum\limits_{k=1}^{n} \alpha_k \neq 1$. Then $E_G = \mathbb{C}^n$.

Lemma 2.4. *Let* G *be a non abelian subgroup of* $\mathcal{H}(n, \mathbb{C})$. *Then* Λ_G *is a subgroup of* $\mathbb{C}^*$. *Moreover,* $0 \in \overline{\Lambda_G}$ *if* $G\backslash\mathcal{R}_n \neq \emptyset$.

Proof. One has $1 \in \Lambda_G$ since $id_{\mathbb{R}^n} \in G$. Let $\lambda, \mu \in \Lambda_G$ and $f, g \in G$ defined by $f : x \longmapsto \lambda x + a$, and $g : x \longmapsto \mu x + b$, $x \in \mathbb{R}^n$, so $f \circ g^{-1}(x) = \frac{\lambda}{\mu} x - \frac{\lambda b}{\mu} + a$. Hence $\frac{\lambda}{\mu} \in \Lambda_G$. Moreover, $\Lambda_G \backslash S^1 \neq \emptyset$, if $G\backslash\mathcal{R}_n \neq \emptyset$. So $\underset{m \to \pm\infty}{lim} \alpha^m = 0$, for any $\alpha \in \Lambda_G \backslash S^1$. It follows that $0 \in \overline{\Lambda_G}$. □

Lemma 2.5. *Let* G *be a non abelian subgroup of* $\mathcal{H}(n, \mathbb{C})$ *with* $G\backslash\mathcal{SR}_n \neq \emptyset$ *and* $\Lambda_G \backslash \mathbb{R} \neq \emptyset$. *Then:*

(i) $\Lambda_G \backslash (F_2 \cup F_3 \cup \mathbb{R}) \neq \emptyset$.

(ii) *if E_G is a vector space, there exist $f_1 = (a_1, \lambda), \dots, f_p = (a_p, \lambda) \in G \backslash \mathcal{SR}_n$ such that $\lambda \in \Lambda_G \backslash \mathbb{R}$ and $(a_1, \dots, a_p)$ is a basis of E_G.*

(iii) *if $G' = T_{-a} \circ G \circ T_a$ for some $a \in \Gamma_G$, then $E_{G'} = T_{-a}(E_G)$ and $\Lambda_{G'} = \Lambda_G$.*

Proof. (i) Let $f = (a, \lambda'), g = (b, \mu) \in G$ such that $\lambda' \in \Lambda_G \backslash \mathbb{R}$ and $\mu \notin (F_2 \cup F_3)$. Suppose that $\lambda' \in F_i$ for some $i \in \{2, 3\}$ and $\mu \in \mathbb{R}$, so $|\lambda'| = 1$ and $|\mu| \neq 1$ since $-1, 1 \in F_2 \cup F_3$. Then $|\lambda'\mu| \neq 1$, it follows that $\lambda'\mu \notin (F_2 \cup F_3)$ and $\lambda'\mu \notin \mathbb{R}$ because $\mu \in \mathbb{R}$ and $\lambda' \notin \mathbb{R}$. Therefore $f \circ g = (c, \lambda'\mu)$ for $c = \frac{-\lambda'\mu b + \lambda'(b-a)+a}{1-\lambda'\mu}$, so $\lambda = \lambda'\mu \in \Lambda_G \backslash (F_2 \cup F_3 \cup \mathbb{R})$.

(ii) Suppose that $\dim(vect(\Gamma_G)) = k < \dim(E_G) = p$. By Lemma 2.1 (iii), $k \geq 1$. By (i), we let $a_1, \dots, a_k \in \Gamma_G$ and $b_{k+1}, \dots, b_p \in G_1(0)$, such that:
- $\mathcal{B}_1 = (a_1, \dots, a_k, b_{k+1}, \dots, b_p)$ is a basis of E_G.
- $(a_1, \dots, a_k)$ is a basis of $vect(\Gamma_G)$.
- $f = (a_1, \lambda) \in G \backslash \mathcal{SR}_n$ with $\lambda \in \Lambda_G \backslash \mathbb{R}$. (i.e. $\lambda \in \Lambda_G \backslash (F_2 \cup F_3 \cup \mathbb{R})$).
-$f_k = (a_i, \lambda_i) \in G \backslash \mathcal{T}_n$, $i = 2, \dots, k$.

Write $f_i' = f_i \circ f \circ f_i^{-1}$, for every $2 \leq i \leq k$. We have $f_i' = (f_i(a_1), \lambda) \in G \backslash \mathcal{T}_n$. See that $f_i(a_1) = \lambda_i a_1 + (1 - \lambda_i) a_i \in \Gamma_G$, $i = 2, \dots, k$.

For every $k + 1 \leq i \leq p$ there exists $T_i \in G_1$ such that $T_i(0) = b_i$. Write $f_i = T_i \circ f \circ T_i^{-1}$, for every $k+1 \leq i \leq p$. We have $f_i = (T_i(a_1), \lambda) \in G \backslash \mathcal{SR}_n$. So $T_i(a_1) \in \Gamma_G$ and $T_i(a_1) = a_1 + b_i$, $k + 1 \leq i \leq p$.

Let's show that $\mathcal{B}_2 = (f_1(a_1), \dots, f_k(a_1), T_{k+1}(a_1), \dots, T_p(a_1))$ is a basis of E_G: Let

$$M = \begin{bmatrix} B & A \\ 0 & I_{p-k} \end{bmatrix} \in M_n(\mathbb{C}), \quad \text{with} \quad A = \begin{bmatrix} 1 & \dots & 1 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} \in M_{k,p-k}(\mathbb{C}),$$

$$B = \begin{bmatrix} 1 & \lambda_2 & \lambda_3 & \dots & \lambda_k \\ 0 & 1-\lambda_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1-\lambda_k \end{bmatrix} \in M_k(\mathbb{C}).$$

and I_{p-k} is the identity matrix of $M_{p-k}(\mathbb{C})$. As $f_i \in G \backslash \mathcal{T}_n$, $\lambda_i \neq 1$, for every $i = 2, \dots, k$, so M is invertible and $M(\mathcal{B}_1) = \mathcal{B}_2$. So $\mathcal{B}_2$ is a basis of E_G contained in Γ_G, a contradiction. We conclude that $k = p$.

(iii) Suppose that E_G is an affine subspace of $\mathbb{R}^n$ with dimension p. Let $a \in \Gamma_G$ and $G' = T_{-a} \circ G \circ T_a$. Set $f = (a, \lambda) \in G \backslash \mathcal{T}_n$, then $T_{-a} \circ f \circ T_a = \lambda id_{\mathbb{C}^n} \in G' \backslash \mathcal{T}_n$, so $0 \in \Gamma_{G'} \subset E_{G'}$, hence $E_{G'}$ is a vector space. By *(ii)* there exists a basis $(a_1', \dots, a_p')$ of $E_{G'}$ contained in $\Gamma_{G'}$. Since $\Gamma_{G'} = T_{-a}(\Gamma_G)$, we let $a_k = T_a(a_k')$, $1 \leq k \leq p$, then $a_1, \dots, a_p \in \Gamma_G$. We have $\Gamma_{G'} = T_{-a}(\Gamma_G) \subset T_{-a}(E_G)$ and $T_{-a}(E_G)$ is a vector subspace of $\mathbb{R}^n$ with dimension p, containing $a_1', \dots, a_p'$. Therefore $E_{G'} = T_{-a}(E_G)$.

On the other hand, for every $g = (b, \mu) \in G\backslash\mathcal{T}_n$, $T_{-a} \circ g \circ T_a = (b - a, \mu)$, so $\Lambda_{G'} = \Lambda_G$. $\square$

Lemma 2.6. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{C})$. Then:*

(i) *If E_G is a vector subspace of $\mathbb{C}^n$, then $G(0) \subset E_G$.*
(ii) *Γ_G and E_G are G-invariant.*

Proof. (i) By construction, $G_1(0) \subset E_G$. Let $f \in G\backslash G_1$, then $f = (a, \lambda)$, for some $\lambda \in \Lambda_G$ and $a \in \Gamma_G \subset E_G$. Therefore $f(z) = \lambda(z - a) + a$, $z \in \mathbb{C}^n$ and $f(0) = (1 - \lambda)a$, so $f(0) \in E_G$ since E_G is a vector space.

(ii) Γ_G is G-invariant: Let $a \in \Gamma_G$ and $g \in G$ then there exists $\lambda \in \mathbb{C}\backslash(F_2 \cup F_3)$ such that $f = (a, \lambda) \in G\backslash\mathcal{SR}_n$. We let $h = g \circ f \circ g^{-1} = (g(a), \lambda) \in G\backslash\mathcal{SR}_n$, so $g(a) \in \Gamma_G$ and hence Γ_G is G-invariant.
E_G is G-invariant: Let $a \in \Gamma_G$ and $G' = T_{-a} \circ G \circ T_a$. We have G' is a non abelian subgroup of $\mathcal{H}(n, \mathbb{C})$ and $E_{G'} = T_{-a}(E_G)$ is a vector subspace of $\mathbb{C}^n$. Let $f \in G'$ having the form $f(z) = \lambda z + b$, $z \in \mathbb{C}^n$. By (i), $b = f(0) \in \Gamma_{G'} \subset E_{G'}$. So for every $z \in E_{G'}$, $f(z) \in E_{G'}$, hence $E_{G'}$ is G'-invariant. By Lemma 2.5.(iii) one has $E_G = T_{-a}(E_{G'})$ is G-invariant. $\square$

Lemma 2.7. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{C})$. Suppose that E_G is a vector space and there exist $f_1 = (a_1, \lambda), \dots, f_p = (a_p, \lambda) \in G\backslash\mathcal{SR}_n$ with $\lambda \in \Lambda_G\backslash\{0, 1\}$ and $\mathcal{B}_1 = (a_1, \dots, a_p)$ is a bases of E_G. Then there exists $f = (a, \lambda) \in G\backslash\mathcal{SR}_n$ such that $\mathcal{B}_2 = (a_1 - a, \dots, a_p - a)$ is also a basis of E_G.*

Proof. Let $a = f_{p-1}(a_p)$, since $f_{p-1} = (a_{p-1}, \lambda) \in G\backslash\mathcal{SR}_n$ then $a = \lambda a_p + (1 - \lambda)a_{p-1}$. By Lemma 2.6.(ii), Γ_G is G-invariant, so $a \in \Gamma_G$. Let

$$P = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ \lambda - 1 & \dots & \dots & \lambda - 1 & \lambda & \lambda - 1 \\ -\lambda & \dots & \dots & -\lambda & \lambda & 1 - \lambda \end{bmatrix}.$$

Since $\lambda \notin \{0, 1\}$, then $\det(P) = 2\lambda(1 - \lambda) \neq 0$ and P is invertible. We have $P(\mathcal{B}_1) = \mathcal{B}_2$ so $\mathcal{B}_2$ is a basis of $\mathbb{R}^n$. $\square$

By the same proofs of Lemmas 2.8 and 2.1, in [5], we can show the following Lemma:

Lemma 2.8. *Let G be the subgroup of $\mathcal{H}(n, \mathbb{C})$ generated by $f_1 = (a_1, \lambda_1), \dots, f_p = (a_p, \lambda_p) \in \mathcal{H}(n, \mathbb{C})\backslash\mathcal{SR}_n$. Then $E_G = \mathcal{A}ff(\{a_1, \dots, a_p\})$.*

Lemma 2.9. *Let G be a subgroup of $\mathcal{H}(n, \mathbb{C})$ generated by $f = (a, \lambda)$ and $g = (b, \mu)$. Then $\Delta = \mathbb{C}(b - a) + a$ is G-invariant and $G_{/\Delta}$ is a subgroup of $\mathcal{H}(1, \mathbb{C})$.*

Proof. Let $\alpha \in \mathbb{C}$, and $z = \alpha(b-a)+a$ we have

$$\begin{aligned} f(z) &= \lambda(\alpha(b-a)+a-a)+a \\ &= \lambda\alpha(b-a)+a \end{aligned} \qquad \text{and} \qquad \begin{aligned} g(z) &= \mu(\alpha(b-a)+a-b)+b \\ &= \mu(\alpha-1)(b-a)+b-a+a. \\ &= (1+\mu(\alpha-1))(b-a)+a \end{aligned}$$

So $f(z),\ g(z) \in \mathbb{C}(b-a)+a$. □

Lemma 2.10. *Let G be the group generated by $h = \lambda Id_{\mathbb{C}}$ and $f = (a, \lambda)$ with $a \in \mathbb{C}^*$, $\lambda \notin \mathbb{C}\backslash\{0,1\}$. Then for every $k \in \mathbb{Z}^*$, one has:*
(i) $\Lambda_G = \{\lambda^j,\ \ j \in \mathbb{Z}\}$ and for every $b \in \Gamma_G$, $g = (b, \lambda) \in G$.
(ii) $(\lambda^k-1)^2 G_1(0) \subset G_1(0)$ and $(\lambda^k-1)^2\Gamma_G \subset G_1(0)$.
(iii) $\lambda^k G_1(0) \subset G_1(0)$.
(iv) if $\lambda^k \neq 1$, $\left(\frac{1}{1-\lambda^k}\right) G_1(0) \subset \Gamma_G$ and $(1-\lambda^k)\Gamma_G \subset \Gamma_G$.

Proof. Let $k \in \mathbb{Z}^*$ such that $\lambda^k \neq 1$.
(i) Let $b \in \Gamma_G$ and $g_1 = (b, \mu) \in G\backslash\mathcal{T}_1$, so $g_1 = h^{n_1} \circ f^{m_1} \circ \dots \circ h^{n_p} \circ f^{m_p}$ for some $p \in \mathbb{N}$ and $n_1, m_1, \dots, n_p, m_p \in \mathbb{Z}$. Then

$$g_1(z) = \lambda^{n_1}\left(\lambda^{m_1}\left(\dots(\lambda^{n_p}(\lambda^{m_p}(z-a)+a)\dots-a\right)+a\right), \quad z \in \mathbb{C}$$

It follows that $\mu = \lambda^j$ with $j = n_1+m_1+\dots+n_p+m_p$ and so $\Lambda_G = \{\lambda^j,\ \ j \in \mathbb{Z}\}$. It follows that $g = (b, \lambda) = g_1^{-j+1} \in G$.

(ii) Let $a \in G_1(0)$ and $g = T_a \circ h \circ T_{-a}$, so $g = (a, \lambda)$ and $g^k \circ h^k \circ g^{-k} \circ h^{-k}(z) = z + (1-\lambda^k)^2 a$, $z \in \mathbb{C}$. So $(\lambda^k-1)^2 a \in G_1(0)$.

Let $b \in \Gamma_G$. By (i), $g = (b, \lambda) \in G$ then $g^k \circ h^k \circ g^{-k} \circ h^{-k}(z) = z + (1-\lambda^k)^2 b$, $z \in \mathbb{C}$. So $(1-\lambda^k)^2 b \in G_1(0)$.

(iii) Let $b \in G_1(0)$. We have $T_b \in G_1$ and

$$\begin{aligned} h^k \circ T_b \circ h^{-k}(z) &= \lambda^k(\lambda^{-k}z+b) \\ &= z+\lambda^k b, \end{aligned}$$

then $\lambda^k b \in G_1(0)$

(iv) Let $a \in G_1(0)$. We have $T_a \in G_1$ and

$$\begin{aligned} T_a \circ h^k(z) &= \lambda^k z + a \\ &= \lambda^k\left(z - \frac{a}{1-\lambda^k}\right) + \frac{a}{1-\lambda^k}, \end{aligned}$$

then $f_1 = T_a \circ h^k = \left(\frac{a}{1-\lambda^k}, \lambda^k\right)$, so

$$\frac{a}{1-\lambda^k} \in \Gamma_G \qquad (1).$$

Let $b \in \Gamma_G$. By (ii), $(1-\lambda^k)^2 b \in G_1(0)$ and by (1), $\frac{(1-\lambda^k)^2 b}{1-\lambda^k} = (1-\lambda^k)b \in \Gamma_G$. □

Lemma 2.11. *Let G be a subgroup of $\mathcal{H}(1,\mathbb{C})$ with $0\in\Gamma_G$. Then:*
(i) $\Lambda_G z+G_1(0)\subset G(z)\subset\Lambda_G z+G(0)$ for every $z\in\mathbb{C}$.
(ii) $(1-\lambda)\Gamma_G\cup G_1(0)\subset G(0)\subset G_1(0)\cup\Gamma_G$.

Proof. Let $h=\lambda id_{\mathbb{C}^n}\in G$ for some $\lambda\in\Lambda_G$, since $0\in\Gamma_G$ and let $f'\in G$ with $f'\circ h\neq h\circ f'$, so $f=f'\circ h\circ f'^{-1}=(a,\lambda)$ for some $a\in\Gamma_G$.

Proof of (i): Let $g=(b,\mu)\in G$, so

$$g(z)=\begin{cases}\mu(z-b)+b=\mu z+(1-\mu)b, & if\ g\in G\backslash\mathcal{T}_1\\ z+b, & if\ g\in G_1\end{cases}\qquad(2)$$

By (2), $b,(1-\mu)b\in G(0)$, so $G(z)\subset\Lambda_G z+G(0)$. Conversely, let $\mu\in\Lambda_G$, $a\in G_1(0)$ so $T_a\in G$. By Lemma 2.10,(iv), $a'=\frac{a}{1-\mu}\in\Gamma_G$ and by Lemma 2.10,(i), $g=(a',\mu)\in G\backslash\mathcal{T}_1$. Then $g(z)=\mu(z-a')+a'=\mu z+(1-\mu)a'$, thus $g(z)=\mu z+a\in G(z)$. It follows that $\Lambda_G z+G_1(0)\subset G(z)$.

Proof of (ii): Let $b\in G(0)$, so $b=f(0)$, for some $f=(a,\mu)\in G$. By (2), $b=a\in G_1(0)$ if $f\in G_1$ and $a\in\Gamma_G$ if $f\in G\backslash\mathcal{T}_1$. By Lemma 2.10,(i), $\mu=\lambda^k\neq 1$ for some $k\in\mathbb{Z}^*$, then $b=(1-\lambda^k)a$ and by Lemma 2.10,(iv), $b\in\Gamma_G$. It follows that $G(0)\subset G_1(0)\cup\Gamma_G$.
Let $b\in\Gamma_G$. By Lemma 2.10,(i), $g=(b,\lambda)\in G\backslash\mathcal{T}_1$, so $g(0)=(1-\lambda)b\in G(0)$. Then $(1-\lambda)\Gamma_G\subset G(0)$. As $G_1(0)\subset G(0)$, the results follows. □

Notice that the following Lemma is a consequence of Theorems 2.1 and 3.1 given in [4], for a closed subgroup of $\mathbb{R}^n$, by identifying $\mathbb{C}^n$ to $\mathbb{R}^{2n}$, we obtain:

Lemma 2.12. *Let H be a closed subgroup of $\mathbb{C}$. Then:*
(1) If H is discrete then $H=\mathbb{Z}a$ or $H=\mathbb{Z}a+\mathbb{Z}b$, for some basis (a,b) of $\mathbb{C}$ over $\mathbb{R}$.
(2) If H is not discrete then there is one of the following:

- (i) $H=\mathbb{C}$.
- (ii) $H=\mathbb{R}a$, *for some* $a\in\mathbb{C}$.
- (iii) $H=\mathbb{R}a+\mathbb{Z}b$, *for some basis* (a,b) *of* $\mathbb{C}$ *over* $\mathbb{R}$.

3. Some results for the case n=1

In this section, we study the case when $n=1$ and G is generated by $f=(a,\lambda)$ and $g=(b,\mu)$ for some $\lambda\in\mathbb{C}\backslash\mathbb{R}$, $\mu\in\mathbb{C}$ and $a,b\in\mathbb{C}^n$ with $a\neq b$.

3.1. Case: $|\lambda|\neq 1$.

Lemma 3.1. *Let $\lambda\in\mathbb{C}\backslash S^1$, $\mu\in\mathbb{C}$ and $a,b\in\mathbb{C}^n$ with $a\neq b$. If G is the group generated by $f=(a,\lambda)$ and $g=(b,\mu)$ then $\overline{G_1(z)}=\mathbb{C}(b-a)+a$ for every $z\in\mathbb{C}(b-a)+a$. In particular, $\overline{G(z)}=\mathbb{C}(b-a)+a$ for every $z\in\mathbb{C}(b-a)+a$.*

Proof. We can assume that $\mu = \lambda$, otherwise we replace g by $g \circ f \circ g^{-1}$ and so G will be the group generated by $f = (a, \lambda)$ and $g = (b, \lambda)$. Suppose that $|\lambda| > 1$ (leaving to replace f by f^{-1}).
(i) Firstly, we will show that G_1 is not discrete. Denote by $G' = T_{-a} \circ G \circ T_a$, then G' is generated by $h = T_{-a} \circ f \circ T_a$ and $g' = T_{-a} \circ g \circ T_a$. We obtain $h = \lambda.id_{\mathbb{C}^n}$ and $g' = (b - a, \lambda)$.

Therefore $h^k \circ g'^k \circ h^{-k} \circ g'^{-k}(z) = z - (\lambda^k - 1)^2(b - a)$, $z \in \mathbb{C}^n$ for every $k \in \mathbb{Z}$. Write $T_{a_k} = h^k \circ g'^k \circ h^{-k} \circ g'^{-k}$ is the translation by $a_k = -(\lambda^k - 1)^2(b - a)$. One has

$$\begin{aligned} a_k - a_{k+1} &= \left((\lambda^{k+1} - 1)^2 - (\lambda^k - 1)^2\right)(b - a) \\ &= \lambda^k(\lambda - 1)(\lambda^{k+1} + \lambda^k - 2)(b - a). \end{aligned}$$

Since $|\lambda| > 1$, it follows that

$$\underset{k \to -\infty}{lim} \|a_k - a_{k+1}\| = 0. \qquad (1)$$

so $G'_1(0)$ can not be discrete.

(ii) Secondly, suppose that $\overline{G'_1(0)} \neq \mathbb{C}(b - a)$, then by (i) and Lemma 2.12, there are two cases:
• Suppose that $\overline{G'_1(0)} = (\mathbb{R}\alpha + \mathbb{Z}\beta)(b - a)$, for some basis $(\alpha(b - a), \beta(b - a))$ of $\mathbb{C}(b - a)$ over $\mathbb{R}$. Let $u = \beta(b - a)$ and T_u the translation by u. See that $u \in \overline{G'_1(0)} \subset \overline{G'(0)}$ and so $G'(u) \subset \overline{G'(0)}$. Remark that $T_u \in \overline{G'_1}$, where $\overline{G'_1}$ is the closure of G'_1 in $\mathcal{T}_n$, then $g_1 = T_{-u} \circ h \circ T_u \in \overline{G'}$, so $g_1 = (u, \lambda)$. Let $b_k = -(\lambda^k - 1)^2 u$ and T_{b_k} be the translation by b_k, $k \in \mathbb{Z}$. As above, we have $T_{b_k} = h^k \circ g_1^k \circ h^{-k} \circ g_1^{-k} \in G'_2 \cap \mathcal{T}_n$, since h and $g_1 \in G'_2$, for every $k \in \mathbb{Z}$. Therefore, by (1), $\underset{k \to -\infty}{lim} \|b_k - b_{k+1}\| = 0$, so $\|b_{k_0} - b_{k_0+1}\| < \frac{1}{2}$, for some $k_0 \in \mathbb{Z}$. Let $v = b_{k_0} - b_{k_0+1}$ then $v = \beta\left((\lambda^{k_0+1} - 1)^2 - (\lambda^{k_0} - 1)^2\right)(b - a) \in \beta\mathbb{R}(b - a)$, so $v \notin (\alpha\mathbb{R} + \beta\mathbb{Z})(a - b) = \overline{G'_1(0)}$ since $\|v\| < \frac{1}{2}$, a contradiction, because $v = T_{b_{k_0}} \circ T_{b_{k_0+1}}(0) \in (G'_2 \cap \mathcal{T}_n)(0) \subset \overline{G'_1(0)}$.
• Suppose that $\overline{G'_1(0)} = \alpha\mathbb{R}(a - b)$, for some $\alpha \in \mathbb{C}^*$. As $\lambda \in \mathbb{C}\backslash\mathbb{R}$, so $\alpha\mathbb{R}(a - b)$ can not be invariant by h. On the other hand, $G'_1(0) \subset \alpha\mathbb{R}(a - b)$, then for any $T_v \in G'_1$, one has $v \in \alpha\mathbb{R}(a - b)$, so $T' = h \circ T_v \circ h^{-1} = T_{h(v)} \in G'_1$, hence $h(v) = \lambda v \in \alpha\mathbb{R}(a - b)$, a contradiction.

(iii) Finally, we conclude that $\overline{G'_1(0)} = \mathbb{C}(a-b)$ and by Lemma 2.9, $\overline{G'(0)} = \mathbb{C}(a-b)$. It follows that $\overline{G(a)} = T_b(\mathbb{C}(a - b)) = \mathbb{C}(b - a) + a$. □

3.2. **Case: $|\lambda| = 1$.** In this case, write $\lambda = e^{i\theta}$, $\theta \in \mathbb{R}$. We identify $\mathbb{C}$ to $\mathbb{R}^2$, by the isomorphism $\varphi : z = x + iy \longrightarrow (x, y)$. State the following results:

Theorem 3.2. *Let G be a group generated by $h = e^{i\theta}Id_1$ and $f = (a, e^{i\theta})$, $a \in \mathbb{C}^*$. Then there is one of the following:*
(i) Every orbit of G is dense in $\mathbb{C}$. In this case $f \notin \mathcal{SR}_1$.
(ii) Every orbit of G is closed and discrete. In this case $f \in \mathcal{SR}_1$.

Proposition 3.3. *If $\theta \notin \pi\mathbb{Q}$, $\mu \in \mathbb{C}$ and $a, b \in \mathbb{C}^n$, with $a \neq b$. If G is the group generated by $f = (a, e^{i\theta})$ and $g = (b, \mu)$ then $\overline{G(a)} = \mathbb{C}(b-a) + a$.*

Proof. By Lemma 2.9, $\Delta = \mathbb{C}(b-a) + a$ is G-invariant and $G_{/\Delta}$ is a subgroup of $\mathcal{H}(1, \mathbb{C})$.
First, we can assume that $\mu = e^{i\theta}$, otherwise we replace g by $g \circ f \circ g^{-1}$, second we suppose that $a = 0$, otherwise we replace G by $T_a \circ G \circ T_{-a}$. Then we will show that $\overline{G(0)} = \mathbb{C}$.
Let $G' = \varphi \circ G \circ \varphi^{-1}$, then G' is the group generated by $R_1 = \varphi \circ h \circ \varphi^{-1}$ and $R_2 = \varphi \circ f \circ \varphi^{-1}$. By a simple calculus, we can check that $R_1 = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ and $R_2 = T_{\varphi(b)} \circ R_1 \circ T_{-\varphi(b)}$ is the rotation with center $\varphi(b)$ and angle θ. Notice by $\|.\|$ the Euclidean norm on $\mathbb{R}^2$ defined by $\|(x, y)\| = \sqrt{x^2 + y^2}$. Let $u = (x_0, y_0) \in \mathbb{R}^2$ and $o = (0, 0)$. There are three cases:

(1) Suppose that $u \neq o$. Write the closed ball $D = \{v \in \mathbb{R}^2, \quad \|v\| < \|u\|\}$ and its border $C = \{v \in \mathbb{R}^2, \quad \|v\| = \|u\|\}$.
(i) Firstly, we will prove that $o \in \mathbb{R}^2 \backslash T(D)$ for some $T \in G'_1$. For every $z \in \mathbb{C}^n$, on has

$$\begin{aligned} h \circ f \circ h^{-1} \circ f^{-1}(z) &= e^{i\theta}(e^{-i\theta}[e^{i\theta}(e^{-i\theta}(z-a) + a) - a] + a) \\ &= z + \left(1 - e^{2i\theta}\right) a. \end{aligned}$$

Write $c = \left(1 - e^{2i\theta}\right) a$, hence $h \circ f \circ h^{-1} \circ f^{-1} = T_c \in G \backslash \{id_{\mathbb{C}}\}$ since $\theta \notin \pi\mathbb{Q}$. Then $T_{\varphi(c)} = \varphi \circ T_c \circ \varphi^{-1} \in G'$, so $T_{n\varphi(c)}(o) \in \mathbb{R}^2 \backslash D$, for some $n \in \mathbb{N}$, we take $T = T_{n\varphi(c)}$.

(ii) Secondly, let's prove that $T(D) \subset \overline{G'(u)}$. Let $b \in T(D)$ and set $C_b = \{v \in \mathbb{R}^2, \quad \|v\| = \|b\|\}$. By (i), $o \notin T(D)$ then $C_b \cap T(C) \neq \emptyset$. Let $b' \in C_b \cap T(C)$, therefore $b' \in \overline{G'(u)}$, because $T \in G'$ and the orbit of u by R_1 is dense in C, since $\theta \notin \pi\mathbb{Q}$, so $C \subset \overline{G'(u)}$. In the same way, one has $C_b \subset \overline{G'(b')} \subset \overline{G'(u)}$, by R_1. It follows that $b \in C_b \subset \overline{G'(u)}$ and so $T(D) \subset \overline{G'(u)}$.

(iii) Finally, we conclude that $G'(u)$ is locally dense for every $u \neq o$.

(2) Suppose that $u = o$, so $R_2(o) \neq o$, by applying (1) on $v = R_2(o)$, we obtain $G'(v)$ is locally dense, so $G'(o)$ is locally dense, since $G'(o) = G'(v)$.

(3) We conclude that every orbit of G' is dense in $\mathbb{R}^2$, since $\mathbb{R}^2$ is connected and every orbit is locally dense. It follows that, every orbit of G is dense in $\mathbb{C}$. □

Lemma 3.4. *Let $\theta \in \pi(\mathbb{Q} \backslash \mathbb{Z})$, $\mu \in \mathbb{C}$ and $a, b \in \mathbb{C}^n$, with $a \neq b$. If G is the group generated by $f = (a, e^{i\theta})$ and $g = (b, \mu)$ then every orbit of G is dense in $\mathbb{C}(b-a)+a$ or is closed and discrete.*

Proof. By Lemma 2.9, $\Delta = \mathbb{C}(b-a) + a$ is G-invariant and $G_{/\Delta}$ is a subgroup of $\mathcal{H}(1, \mathbb{C})$. Firstly, we can assume that $\mu = e^{i\theta}$, otherwise we replace g by $g \circ f \circ g^{-1}$, secondly we suppose that $a = 0$, otherwise we replace G by $T_a \circ G \circ T_{-a}$. Then we will show every orbit of G is dense in $\mathbb{C}$ or closed and discrete.

Thirdly, we will show that $G_1(0)$ is dense in $\mathbb{C}$ or it is closed discrete. Suppose that $\overline{G_1(0)} \neq \mathbb{C}$ and $G_1(0)$ is not discrete. Then by Lemma 2.12, $\overline{G_1(0)} = \mathbb{Z}a_1 + \mathbb{R}a_2$ for some $a_1, a_2 \in \mathbb{R}$ with $a_2 \neq 0$. So $T_{a_1} \in \overline{G_1}$ where $\overline{G}$ be the closure of G in $\mathcal{T}_n$. Let $g = T_{a_1} \circ h \circ T_{-a_1}$, then $g = (a_1, e^{i\theta})$. Since $\theta \in \pi(\mathbb{Q}\backslash\mathbb{Z})$, so $e^{i\theta}\mathbb{R}a_2 \neq \mathbb{R}a_2$. By Lemma 2.10.(iii), $e^{i\theta}\mathbb{R}a_2 \subset \overline{G_1(0)} = \mathbb{Z}a_1 + \mathbb{R}a_2$, a contradiction.

We conclude that $G_1(0)$ is dense in $\mathbb{C}$ or closed and discrete.
Finally, by Lemma 2.10, (iii), (iv) and Lemma 2.11,(i) and (ii) we have $G_1(0)$ is closed discrete or dense if and only if are Γ_G and $G_1(0)$ and this is equivalent to is $G(0)$. On the other hand, $\theta \in \pi(\mathbb{Q}\backslash\mathbb{Z})$, so by Lemma 2.10.(i), Λ_G is finite and the proof results from Lemma 2.11, (i) and (ii). $\square$

Proposition 3.5. *Let $\theta \in \pi(\mathbb{Q}\backslash\mathbb{Z})$, $\mu \in \mathbb{C}$ and $a, b \in \mathbb{C}^n$, with $a \neq b$. If G is the group generated by $f = (a, e^{i\theta})$ and $g = (b, \mu)$ then $G(a)$ is closed and discrete if and only if $G \subset \mathcal{S}_2\mathcal{R}_n$ or $G \subset \mathcal{S}_3\mathcal{R}_n$.*

To prove Proposition 3.5, we need to introduce the following Lemmas:

Lemma 3.6. *Let G be the group generated by $h = e^{i\theta}Id_{\mathbb{C}}$ and $f = (a_0, e^{i\theta})$ with $a_0 \in \mathbb{C}^*$ and $\theta \in \mathbb{R}$. If $G_1(0) = \mathbb{Z}a_1 + \mathbb{Z}a_2$ where (a_1, a_2) is a basis of $\mathbb{C}$ over $\mathbb{R}$ then there exists $P \in GL(2, \mathbb{C})$ such that $Pe_1 = a_1$, $Pe_2 = a_2$ and $P^{-1}R_\theta P \in SL(2, \mathbb{Z})$, where $R_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$.*

Proof. If $G_1(0) = \mathbb{Z}a_1 + \mathbb{Z}a_2$ with (a_1, a_2) is a basis of $\mathbb{C}$ over $\mathbb{R}$. By Lemma 2.10.(iii), $e^{i\theta}a_1, e^{i\theta}a_2 \in G_1(0)$, so

$$\begin{cases} e^{i\theta}a_1 = na_1 + ma_2 \qquad \text{and} \\ e^{i\theta}a_2 = n'a_1 + m'a_2 \end{cases}$$

for some $n, m, n', m' \in \mathbb{Z}$. Write $a_1 = a + ic$ and $a_2 = b + id$, $a, c, b, d \in \mathbb{R}$ then:

$$\begin{cases} (cos\theta + i.sin\theta)(a + ic) = n(a + ic) + m(b + id) \\ (cos\theta + i.sin\theta)(b + id) = n'(a + ic) + m'(b + id) \end{cases}$$

So

$$\begin{cases} a.cos\theta - c.sin\theta = na + mb \\ a.sin\theta + c.cos\theta = nc + md \end{cases} \quad \text{and} \quad \begin{cases} b.cos\theta - d.sin\theta = n'a + m'b \\ b.sin\theta + d.cos\theta = n'c + m'd \end{cases} \qquad (1)$$

Write $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then (1) is equivalent to

$$R_\theta[a, c]^T = P[n, m]^T \quad and \quad R_\theta[b, d]^T = P[n', m']^T.$$

As $Pe_1 = [a, c]^T$ and $Pe_2 = [b, d]^T$, so

$$R_\theta Pe_1 = P[n, m]^T \quad and \quad R_\theta Pe_2 = P[n', m']^T.$$

As (a_1, a_2) is a basis of $\mathbb{C}$ over $\mathbb{R}$ one has $P \in GL(2, \mathbb{R})$ and so

$$P^{-1}R_\theta Pe_1 \in \mathbb{Z}^2 \quad and \quad P^{-1}R_\theta Pe_2 \in \mathbb{Z}^2.$$

It follows that $P^{-1}R_\theta P \in SL(2, \mathbb{Z})$. $\square$

Lemma 3.7. *Let G be the subgroup of $\mathcal{H}(1,\mathbb{C})$ generated by $h = e^{i\theta}Id_{\mathbb{C}}$ and $f = (a, e^{i\theta})$ with $a \in \mathbb{C}^*$ and $\theta \in H_2 \cup H_3$. Then*

$$\big(\mathbb{Z}(1-e^{-i\theta})^2 + \mathbb{Z}(1-e^{i\theta})^2\big)\, a \subset G(0) \subset \big(\mathbb{Z}(1-e^{-i\theta}) + \mathbb{Z}(1-e^{i\theta})\big)\, a.$$

Proof. Denote by $a_1 = (1-e^{-i\theta})a$ and $a_2 = (1-e^{i\theta})a$.
• Firstly, we will prove that $\mathbb{Z}a_1 + \mathbb{Z}a_2$ is G-invariant:
- If $\theta \in H_2$, suppose that $\theta = \frac{\pi}{2}$. Then $e^{-i\theta} = -i$, $e^{i\theta} = i$, so $a_1 = (1-i)a$ and $a_2 = (1+i)a$. Let $u = na_1 + ma_2$, for some $n, m \in \mathbb{Z}$, so

$$\begin{aligned} h(u) &= i(n(1-i)a + m(1+i))a \\ &= na_2 - ma_1 \end{aligned}$$

and

$$\begin{aligned} f(u) &= i\,(n(1-i)a + m(1+i)a - a) + a \\ &= n(1+i)a - m(i-1)a + (1-i)a \\ &= na_2 - (m-1)a_1 \end{aligned}$$

Then $h(u), f(u) \in \mathbb{Z}a_1 + \mathbb{Z}a_2$. It follows that $\mathbb{Z}a_1 + \mathbb{Z}a_2$ is G-invariant.

- If $\theta \in H_3$, suppose that $\theta = \frac{3\pi}{2}$. Then $e^{-i\theta} = e^{-i\frac{\pi}{3}}$ and $e^{i\theta} = e^{i\frac{\pi}{3}}$. As $1-e^{i\frac{\pi}{3}} = e^{-i\frac{\pi}{3}}$ and $1-e^{-i\frac{\pi}{3}} = e^{i\frac{\pi}{3}}$ so $a_1 = e^{i\frac{\pi}{3}}a$ and $a_2 = e^{-i\frac{\pi}{3}}a$. Let $u = na_1 + ma_2$, for some $n, m \in \mathbb{Z}$, so

$$\begin{aligned} h(u) &= e^{\frac{i\pi}{3}}(ne^{i\frac{\pi}{3}}a + me^{-\frac{i\pi}{3}}a) \\ &= ne^{\frac{2i\pi}{3}}a - ma \\ &= -ne^{\frac{-i\pi}{3}}a - m(e^{i\frac{\pi}{3}} + e^{-\frac{i\pi}{3}})a \\ &= (-n-m)e^{\frac{-i\pi}{3}}a - me^{i\frac{\pi}{3}}a \end{aligned}$$

and

$$\begin{aligned} f(u) &= e^{\frac{i\pi}{3}}\left((ne^{i\frac{\pi}{3}}a + me^{-\frac{i\pi}{3}}a - a\right) + a \\ &= ne^{\frac{2i\pi}{3}}a - (m-1)a - e^{\frac{i\pi}{3}}a \\ &= -ne^{\frac{-i\pi}{3}}a - (m-1)(e^{\frac{i\pi}{3}} + e^{-\frac{i\pi}{3}})a - e^{\frac{i\pi}{3}}a \\ &= (-n-m+1)e^{\frac{-i\pi}{3}}a - me^{\frac{i\pi}{3}}a \end{aligned}$$

Then $h(u), f(u) \in \mathbb{Z}a_1 + \mathbb{Z}a_2$. It follows that $\mathbb{Z}a_1 + \mathbb{Z}a_2$ is G-invariant.
• Secondly, $G(0) \subset \mathbb{Z}a_1 + \mathbb{Z}a_2$, since $0 \in \mathbb{Z}a_1 + \mathbb{Z}a_2$ and by above, $\mathbb{Z}a_1 + \mathbb{Z}a_2$ is G-invariant. In particular

$$G_1(0) \subset G(0) \subset \mathbb{Z}a_1 + \mathbb{Z}a_2 \qquad (1).$$

• Finally, by Lemma 2.10.(iii), $(1-e^{-i\theta})^2 a, (1-e^{i\theta})^2 a \in G_1(0)$ since $a \in \Gamma_G$. As $G_1(0)$ is an additive group then

$$\mathbb{Z}(1-e^{-i\theta})^2 a + \mathbb{Z}(1-e^{i\theta})^2 a \subset G_1(0) \subset G(0) \qquad (2).$$

□

Lemma 3.8. *Let G be the subgroup of $\mathcal{H}(1,\mathbb{C})$ generated by $h = \lambda Id_{\mathbb{C}}$ and $f = (a,\lambda)$ with $a \in \mathbb{C}^*$, $\lambda \notin \mathbb{R}$. If $G_1(0)$ is discrete then $G_1(0) = \mathbb{Z}a_1 + \mathbb{Z}a_2$ for some basis (a_1, a_2) of $\mathbb{C}$ over $\mathbb{R}$.*

Proof. By Lemma 2.10.(ii), $0 \neq (\lambda-1)^2 a \in G_1(0)$. Write $a_1 = (\lambda-1)^2 a$. By Lemma 2.10.(iii), $a_2 = \lambda a_1 \in G_1(0)$. As $\lambda \notin \mathbb{R}$, (a_1, a_2) is a basis of $\mathbb{C}$ over $\mathbb{R}$. Then $\mathbb{Z}a_1 + \mathbb{Z}a_2 \subset G_1(0)$ since $G_1(0)$ is an additive group. By Lemma 2.12, $G_1(0) = \mathbb{Z}a'_1 + \mathbb{Z}a'_2$ for some basis (a'_1, a'_2) of $\mathbb{C}$ over $\mathbb{R}$. □

Proof of Proposition 3.5. By Lemma 2.9, $\Delta = \mathbb{C}(b-a)+a$ is G-invariant and $G_{/\Delta}$ is a subgroup of $\mathcal{H}(1,\mathbb{C})$.
First, we can assume that $\mu = e^{i\pi\theta}$, otherwise we replace g by $g \circ f \circ g^{-1}$, second we suppose that $a = 0$, leaving to replace G by $T_a \circ G \circ T_{-a}$. Then we will show that $G(0)$ is closed and discrete if and only if $G \subset \mathcal{S}_2\mathcal{R}_n$ or $G \subset \mathcal{S}_3\mathcal{R}_n$. Then G is generated by $h = e^{i\theta} Id_{\mathbb{C}}$ and $g = (b, e^{i\theta})$ with $b \in \mathbb{C}^*$ and $\theta \in \mathbb{R}$. If $G(0)$ is discrete so is $G_1(0)$. Therefore, by Lemma 3.8, $G_1(0) = \mathbb{Z}a_1 + \mathbb{Z}a_2$ for some basis (a_1, a_2) of $\mathbb{C}$ over $\mathbb{R}$. By Lemma 3.6, there exists $P \in GL(2,\mathbb{R})$ such that $P^{-1}R_\theta P \in SL(2,\mathbb{Z})$, where $R_\theta = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$. Write $P = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ and $A = P^{-1}R_\theta P$, so

$$A = \begin{bmatrix} cos\theta - \left(\frac{a'b'+d'c'}{a'd'-b'c'}\right) sin\theta & \left(\frac{b^2+d^2}{a'd'-b'c'}\right) sin\theta \\ \\ -\left(\frac{a'^2+c'^2}{a'd'-b'c'}\right) sin\theta & cos\theta + \left(\frac{a'b'+d'c'}{a'd'-b'c'}\right) sin\theta \end{bmatrix}.$$

As $A \in SL(2,\mathbb{Z})$ then there exist $n, m \in \mathbb{Z}$ such that

$$\begin{cases} cos\theta - \left(\frac{a'b'+d'c'}{a'd'-b'c'}\right) sin\theta = n \\ cos\theta + \left(\frac{a'b'+d'c'}{a'd'-b'c'}\right) sin\theta = m \end{cases}$$

so $cos\theta = \frac{n+m}{2} \in \frac{1}{2}\mathbb{Z}$. Hence $cos\theta \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$ since $\theta \notin \pi\mathbb{Z}$, therefore $sin\theta \in \left\{-1, -\frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}\right\}$. Thus $\theta \in (\frac{\pi}{2} + \pi\mathbb{Z}) \cup (-\frac{\pi}{3} + \pi\mathbb{Z}) \cup (\frac{\pi}{3} + \pi\mathbb{Z})$. Then $G \subset \mathcal{S}_2\mathcal{R}_n$ if $\theta \in (\frac{\pi}{2} + \pi\mathbb{Z})$ and $G \subset \mathcal{S}_3\mathcal{R}_n$ if $\theta \in (-\frac{\pi}{3} + \pi\mathbb{Z}) \cup (\frac{\pi}{3} + \pi\mathbb{Z})$. The converse follows from Lemma 3.4. The proof is complete. □

3.3. **Proof of Theorem 3.2.**

Lemma 3.9. *$\lambda, \mu \in \mathbb{C}^*$ and $a, b \in \mathbb{C}^n$, with $a \neq b$. If G is the group generated by $f = (a,\lambda)$ and $g = (b,\mu)$ such that $\overline{G(a)} = \mathbb{C}(b-a)+a$ then $\overline{G(z)} = \mathbb{C}(b-a)+a$, for every $z \in \mathbb{C}(b-a)+a$.*

Proof. Let $z \in \mathbb{C}(b-a)+a$. There are three cases:
- If $|\lambda| \neq 1$ or $|\mu| \neq 1$, then by Lemma 3.1, $\overline{G_1(z)} = \mathbb{C}(b-a)+a$, so $\overline{G(z)} = \mathbb{C}(b-a)+a$.
- If $|\lambda| = |\mu| = 1$, then $G \subset \mathcal{R}_n$. Since $z \in \overline{G(a)}$, then there exists a sequence $(g_m)_m \subset G$ such that $\lim\limits_{m\to+\infty} g_m(a) = z$. Write $g_m = (a_m, \varepsilon_m)$, with $|\varepsilon_m| = 1$ for every $m \in \mathbb{N}$. Since $(g_m a)_m$ is bounded then is $(a_m)_m$. Therefore, there is a

subsequence $(a_{\varphi(m)})_m$ such that $\underset{m\to+\infty}{lim} a_{\varphi(m)} = c$ and $\underset{m\to+\infty}{lim} \varepsilon_{\varphi(m)} = \varepsilon$, for some $c \in \mathbb{C}^n$ and $\varepsilon \in S^1$. Moreover, $\underset{m\to+\infty}{lim} g_{\varphi(m)} = h = (c, \varepsilon)$. Therefore $\underset{m\to+\infty}{lim} g_{\varphi(m)}(a) = h(a) = z$. So $\underset{m\to+\infty}{lim} g^{-1}_{\varphi(m)}(z) = h^{-1}(z) = a$. It follows that $a \in \overline{G(z)}$, hence $\mathbb{C}(b-a)+a = \overline{G(a)} \subset \overline{G(z)}$. □

Corollary 3.10. *Let $\theta \in \mathbb{R}$, $\mu \in \mathbb{C}$ and $a, b \in \mathbb{C}^n$, with $a \neq b$. If G is the group generated by $f = (a, e^{i\theta})$ and $g = (b, \mu)$ such that $G \backslash \mathcal{SR}_n \neq \emptyset$ then every orbit of G is dense in $\mathbb{C}(b-a)+a$.*

Proof. The proof results from Lemma 3.1, Proposition 3.3, Lemma 3.4, Proposition 3.5 and Lemma 3.9. □

Lemma 3.11. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{C})$. If $G \backslash \mathcal{R}_n \neq \emptyset$, then for every $z \in \mathbb{C}^n$ we have $\Gamma_G \subset \overline{G(z)}$.*

Proof. Let $z \in \mathbb{C}^n$, $a \in \Gamma_G$ and $f = (a, \lambda) \in G \backslash \mathcal{R}_n$ with $|\lambda| \neq 1$. Suppose that $|\lambda| > 1$ and so

$$\underset{k\longrightarrow-\infty}{lim} f^k(z) = \underset{k\longrightarrow-\infty}{lim} \lambda^k(z-a)+a = a.$$

Hence $a \in \overline{G(z)}$. It follows that $\Gamma_G \subset \overline{G(z)}$. □

Proof of Theorem 3.2: The proof of Theorem 3.2 results from Proposition 3.5 and Corollary 3.10.

Proof of Theorem 1.5: Let $\widetilde{G} = \varphi^{-1} \circ G \circ \varphi$, so $\widetilde{G}$ is a non abelian subgroup of $\mathcal{H}(1, \mathbb{C})$. Firstly, if $(H_2 \cup H_3) \backslash \{\theta, \theta'\} \neq \emptyset$ then $\widetilde{G} \backslash \mathcal{SR}_2 \neq \emptyset$. Therefore:
The proof of (1).(i) results from Theorem 3.2. Let's prove (1).(ii):
Suppose that $\theta \in H_2$ and $\theta' \in H_3$, then by using the analytic form $f = \varphi^{-1} \circ R_\theta \circ \varphi$ (resp. $g = \varphi^{-1} \circ R_{\theta'} \circ \varphi$) of R_θ (resp. $R_{\theta'}$) we have $f = (a, e^{i\theta}) \in \mathcal{S}_2\mathcal{R}_2$ and $g = (b, e^{i\theta'}) \in \mathcal{S}_3\mathcal{R}_2$, where $\varphi(a)$ (resp. $\varphi(b)$) is the center of R_θ (resp. $R_{\theta'}$). Then $f \circ g = \left(c, e^{i(\theta+\theta')}\right)$ with $c = \frac{e^{i\theta}(b-a)-a}{1-e^{i(\theta+\theta')}}$. See that $\theta+\theta' \in \left(\frac{5\pi}{6} + \pi\mathbb{Z}\right) \cup \left(\frac{7\pi}{6} + \pi\mathbb{Z}\right)$. Then $\theta+\theta' \notin H_2 \cup H_2$. The assertion (1).$(ii)$ follows then from (1).(i).
(2) In this case, we can assume that $\widetilde{G} \subset \mathcal{SR}_2$. By Lemma 3.7, if $\widetilde{G} \subset \mathcal{S}_2\mathcal{R}_2$ or $\widetilde{G} \subset \mathcal{S}_3\mathcal{R}_2$ then every orbit is closed and discrete. By (1), it remains to verify the following case: $\theta, \theta' \in H_i$ for some $i \in \{2, 3\}$. Then $\widetilde{G} \subset \mathcal{S}_i\mathcal{R}_2$. The results follows from Lemma 3.7. The proof is complete.

4. Some results in the case $G\backslash\mathcal{SR}_n \neq \emptyset$ for $n \geq 1$

We give some Lemmas and propositions, will be used to prove Theorem 1.1.

Lemma 4.1. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$ such that $G\backslash\mathcal{SR}_n \neq \emptyset$. So $G_1 \neq \{id_{\mathbb{C}^n}\}$ and if $0 \in \Gamma_G$ then $G_1(0) \subset \Gamma_G$.*

Proof. Let $f, g \in G$ such that $f \circ g \neq g \circ f$. Write $f : z \longmapsto \lambda z + a$ and $g : z \longmapsto \mu z + b$. So for every $z \in \mathbb{C}^n$, one has

$$\begin{aligned} f \circ g \circ f^{-1} \circ g^{-1}(z) &= \lambda\left(\mu\left(\frac{1}{\lambda}\left(\frac{1}{\mu}z - \frac{b}{\mu}\right) - \frac{a}{\lambda}\right) + b\right) + a \\ &= z + (\lambda - 1)\, b + (1 - \mu)\, a. \end{aligned}$$

Hence $f \circ g \circ f^{-1} \circ g^{-1} = T_c \in G_1\backslash\{id_{\mathbb{C}^n}\}$, with $c = (\lambda - 1)\, b + (1 - \mu)\, a$.
Suppose now that $0 \in \Gamma_G$, so there $h = \lambda id_{\mathbb{C}^n} \in G\backslash\mathcal{SR}_n$ for some $\lambda \in \Lambda_G$. Let $a \in G_1(0)$, then $T_a \circ h \circ T_{-a} = (a, \lambda) \in G\backslash\mathcal{SR}_n$. So $a \in \Gamma_G$. The proof is complete. □

Proposition 4.2. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$ such that $\Lambda_G\backslash\mathbb{R} \neq \emptyset$ and $G\backslash\mathcal{SR}_n \neq \emptyset$. Then $\overline{G(z)} = E_G$, for every $z \in E_G$.*

To prove the Proposition, we need the following Lemmas:

Lemma 4.3. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$, $f \in G$ and $u, v \in \mathbb{C}^n$ then $f(\mathbb{C}u + v) = \mathbb{C}u + f(v)$.*

Proof. Every $f \in G$ has the form $f(z) = \lambda z + a$, $z \in \mathbb{C}^n$. Let $\alpha \in \mathbb{C}$ then $f(\alpha u + v) = \lambda(\alpha u + v) + a = \lambda\alpha u + (\lambda u + v) = \lambda\alpha u + f(v)$. So $f(\mathbb{C}u + v) \subset \mathbb{C}u + f(v)$, then $f(\mathbb{C}u + v) = \mathbb{C}u + f(v)$. □

Lemma 4.4. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$ such that E_G is a vector subspace of $\mathbb{C}^n$ and $\Gamma_G \neq \emptyset$. Let $a, a_1, \dots, a_p \in \Gamma_G$ such that $(a_1, \dots, a_p)$ and $(a_1 - a, \dots, a_p - a)$ are two basis of E_G and let $D_k = \mathbb{C}(a_k - a) + a$, $1 \leq k \leq p$. If $D_k \subset \overline{G(a)}$ for every $1 \leq k \leq p$, then $\overline{G(a)} = E_G$.*

Proof. The proof is done by induction on $\dim(E_G) = p \geq 1$.
• For $p = 1$, if there exists $a, a_1 \in \Gamma_G$ with $a \neq a_1$ such that $D_1 \subset \overline{G(a)}$, where $D_1 = \mathbb{C}(a_1 - a) + a$, then $\overline{G(a)} = E_G$, since $D_1 = E_G = \mathbb{C}$.
• Suppose that Lemma 4.4 is true until dimension $p - 1$. Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$ with $\Gamma_G \neq \emptyset$ and let $a, a_1, \dots, a_p \in \Gamma_G$ such that $(a_1, \dots, a_p)$ is a basis of E_G. Suppose that $D_k \subset \overline{G(a)}$ for every $1 \leq k \leq p$.
Denote by H the vector subspace of E_G generated by $(a_1 - a), \dots, (a_{p-1} - a)$ and $\Delta_{p-1} = T_a(H)$. We have $\Delta_{p-1} = Aff(a, a_1, \dots, a_{p-1})$.
Set $\lambda, \lambda_k \in \Gamma_G$, $1 \leq k \leq p - 1$ such that $f = (a, \lambda)$, $f_k = (a_k, \lambda_k) \in G\backslash\mathcal{SR}_n$. We let G_k be the group generated by f and f_k for every $1 \leq k \leq p - 1$, so $G_k\backslash\mathcal{SR}_n \neq \emptyset$. By Corollary 3.10, we have $\overline{G_k(a)} = D_k$ for every $k = 1, \dots, p - 1$. Let G' be the subgroup of G generated by f, f_1,…,f_{p-1}, then $D_k \subset \overline{G'(a)}$ for every $1 \leq k \leq p-1$.

By Lemma 2.8 we have $E_{G'} = \Delta_{p-1}$. Let $G'' = T_{-a} \circ G' \circ T_a$, by Lemma 2.5.(iii) we have $E_{G''} = T_{-a}(\Delta_{p-1}) = H$ and $D'_k = T_{-a}(D_k) \subset \overline{G''(0)}$ for every $1 \leq k \leq p-1$. By induction hypothesis applied to G'' we have $\overline{G''(0)} = H$ so $\overline{G'(a)} = \Delta_{p-1}$. Since $G'(a) \subset G(a)$, then

$$\Delta_{p-1} \subset \overline{G(a)} \qquad (1)$$

Let $z \in E_G \backslash \Delta_{p-1}$ and $D = \mathbb{C}(a_p - a) + z$. Since $(a_1 - a, \dots, a_p - a)$ is a basis of E_G, so $H \oplus \mathbb{C}(a_p - a) = E_G$. As $a, z \in E_G$, then $z - a = x + \alpha(a_p - a)$ for some $x \in H$ and $\alpha \in \mathbb{C}$. Let $y = x + a$, as $T_a(H) = \Delta_{p-1}$ we have $y \in \Delta_{p-1}$, and

$$\begin{aligned} y &= x + a \\ &= z - a - \alpha(a_p - a) + a \\ &= -\alpha(a_p - a) + z \in D. \end{aligned}$$

Hence $\quad y \in \Delta_{p-1} \cap D$.

By (1) we have $y \in \overline{G(a)}$. Then there exists a sequence $(f_m)_{m\in\mathbb{N}}$ in G such that $\underset{m\longrightarrow+\infty}{lim} f_m(a) = y$. For every $m \in \mathbb{N}$ denote by $f_m = (b_m, \lambda_m)$.

Remark that $D = \mathbb{C}(a_p - a) + y$, since $z, y \in D$. By Lemma 4.3 we have $f_m(D_p) = f_m(\mathbb{C}(a_p - a) + a) = \mathbb{C}(a_p - a) + f_m(a)$. Since $\underset{m\longrightarrow+\infty}{lim} f_m(a) = y$ then for every $v = \alpha(a_p - a) + y \in D$, $\alpha \in \mathbb{C}$, one has

$$\underset{m\longrightarrow+\infty}{lim} f_m(\alpha(a_p - a) + a) = v.$$

As $\alpha(a_p - a) + a \in D_p \subset \overline{G(a)}$, then $v \in \overline{G(a)}$. Therefore $D \subset \overline{G(a)}$, so $z \in \overline{G(a)}$, hence

$$E_G \backslash \Delta_{p-1} \subset \overline{G(a)} \qquad (2).$$

By (1) and (2) we obtain $E_G \subset \overline{G(a)}$. Since $\Gamma_G \neq \emptyset$ then by Lemma 2.6.(ii), we have E_G is G-invariant, so $G(a) \subset E_G$ since $a \in E_G$. It follows that $\overline{G(a)} = E_G$. $\square$

Proof of Proposition 4.2. Let G be a non abelian subgroup of $\mathcal{H}(n, \mathcal{R})$. Since $G \backslash \mathcal{SR}_n \neq \emptyset$ then $\Gamma_G \neq \emptyset$ and suppose that E_G is a vector subspace of $\mathbb{C}^n$, (one can replace G by $G' = T_{-a} \circ G \circ T_a$, for some $a \in \Gamma_G$).

(i) We will prove that there exists $a \in \Gamma_G$ such that $\overline{G(a)} = E_G$. By Lemmas 2.5,(ii) and 2.7, there exist $f = (a, \lambda), f_1 = (a_1, \lambda), \dots, f_p = (a_p, \lambda) \in G \backslash \mathcal{SR}_n$ such that $\lambda \in \Lambda_G \backslash \mathbb{R}$, $(a_1, \dots, a_p)$ and $(a_1 - a, \dots, a_p - a)$ are two basis of E_G. Denote by $D_k = \mathbb{C}(a_k - a) + a$, $1 \leq k \leq p$. For every $k = 1, \dots, p-1$, we let G_k be the group generated by f and f_k. One has $G_k \backslash \mathcal{SR}_n \neq \emptyset$, so by Corollary 3.10, we have $D_k = \overline{G_k(a)} \subset \overline{G(a)}$ for every $1 \leq k \leq p$. By Lemma 4.4, we have $\overline{G(a)} = E_G$.

(ii) We will prove that $\overline{G(z)} = E_G$ for every $z \in E_G$. By (i) there exists $a \in \Gamma_G$ such that $\overline{G(a)} = E_G$. There are two cases:

- If $G \backslash \mathcal{R}_n \neq \emptyset$, so by Lemma 3.11, $\Gamma_G \subset \overline{G(z)}$. By Lemma 2.6.(ii), Γ_G and E_G are G-invariant, then $\overline{G(a)} \subset \overline{\Gamma_G} \subset \overline{G(z)}$ and so $E_G = \overline{G(z)}$.
- Suppose that $G \subset \mathcal{R}_n$. Since $\overline{G(a)} = E_G$, there exists a sequence $(f_m)_m \subset G$

such that $\underset{m\to+\infty}{lim} f_m(a) = z$. There are two situations:
- $f_m = (a_m, \lambda_m) \in G\backslash\mathcal{T}_n$ for every $m > n_0$, for some $n_0 > 1$. One has $f_m(a) = \lambda_m a + (1-\lambda_m)a_m$ and the sequence $(f_m(a))_m$ is bounded, then the sequence $((1-\lambda_m)a_m)_m$ is bounded and so is $(a_m)_m$. Therefore, there exists a subsequence $(\lambda_{\varphi(m)})_m$ of $(\lambda_m)_m$ and a subsequence $(a_{\varphi(m)})_m$ of $(a_m)_m$ such that $\underset{m\to+\infty}{lim} \lambda_{\varphi(m)} = \lambda'$ and $\underset{m\to+\infty}{lim} a_{\varphi(m)} = b$, for some $b \in E_G$ and $\lambda' \in S^1$, so $\lambda' \neq 0$. Let $f = (b, \lambda')$ if $\lambda' \neq 1$ and $f = T_b$ if $\lambda' = 1$. Therefore $\underset{m\to+\infty}{lim} f_m = f$. Since $\lambda \neq 0$, f is invertible and $\underset{m\to+\infty}{lim} f_m^{-1} = f^{-1}$, so $f \in \overline{G}$. As $\underset{m\to+\infty}{lim} f_m(a) = z = f(a)$, we have $a = f^{-1}(z) = \underset{m\to+\infty}{lim} f_m^{-1}(z)$. It follows that $a \in \overline{G(z)}$, so $E_G = \overline{G(a)} \subset \overline{G(z)} \subset E_G$, since E_G is G-invariant (Lemma 2.6,(ii)).
- $f_m = T_{a_m} \in G \cap \mathcal{T}_n$ for every $m > n_0$, for some $n_0 > 1$. As $\underset{m\to+\infty}{lim} f_m(a) = z$, $\underset{m\to+\infty}{lim} a_m = z - a$, then $\underset{m\to+\infty}{lim} f_m = T_{z-a} \in \overline{G}$. Therefore $a = T_{a-z}(z) = \underset{m\to+\infty}{lim} f_m^{-1}(z)$. It follows that $a \in \overline{G(z)}$, so $E_G = \overline{G(a)} \subset \overline{G(z)} \subset E_G$, since E_G is G-invariant (Lemma 2.6,(ii)). The proof is complete. □

Proposition 4.5. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$. Suppose that $\Lambda_G\backslash\mathbb{R} \neq \emptyset$, $G\backslash\mathcal{SR}_n \neq \emptyset$ and E_G is a vector space. Then for every $z \in \mathbb{C}^n\backslash E_G$, we have $\overline{G(z)} = \overline{\Lambda_G}.z + E_G$.*

To prove the above Proposition, we need the following Lemma:

Lemma 4.6. *Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$ such that $G\backslash\mathcal{SR}_n \neq \emptyset$. For every $b \in E_G$ and for every $\lambda \in \Lambda_G$ there exists a sequence $(f_m)_{m\in\mathbb{N}}$ in G such that $\underset{m\longrightarrow+\infty}{lim} f_m = f = (b, \lambda)$.*

Proof. Let $\lambda \in \Lambda_G$ and $b \in E_G$. Given $g = (a, \lambda) \in G$, so $a \in (\Gamma_G \cup G_1(0)) \subset E_G$. By Proposition 4.2, we have $\overline{G(a)} = E_G$. Then there exists a sequence $(g_m)_{m\in\mathbb{N}}$ in G such that $\underset{m\longrightarrow+\infty}{lim} g_m(a) = b$. For every $m \in \mathbb{N}$, denote by $f_m = g_m \circ g \circ g_m^{-1}$, so $f_m = (g_m(a), \lambda)$. Hence $\underset{m\longrightarrow+\infty}{lim} f_m = f$, with $f = (b, \lambda)$. □

Proof of Proposition 4.5. Let G be a non abelian subgroup of $\mathcal{H}(n,\mathbb{C})$ such that $G\backslash\mathcal{SR}_n \neq \emptyset$ and E_G is a vector space. Let $z \in U = \mathbb{C}^n\backslash E_G$.
Let's prove that $\overline{\Lambda_G}.z + E_G \subset \overline{G(z)}$: Let $\alpha \in \Lambda_G$ and $a \in E_G$.
• Suppose that $\alpha \in \Lambda_G\backslash\{1\}$. Since E_G is a vector space, $a' = \frac{a}{1-\alpha} \in E_G$. By Lemma 4.6 there exists a sequence $(f_m)_m$ in G such that $\underset{m\longrightarrow+\infty}{lim} f_m = f = (a', \alpha) \in G\backslash\mathcal{T}_n$. Then

$$\begin{aligned} f(z) &= \alpha(z - a') + a' \\ &= \alpha z + (1-\alpha)a' \\ &= \alpha z + a \in \overline{G(z)}, \end{aligned}$$

so

$$(\Lambda_G\backslash\{1\})\,.z + E_G \subset \overline{G(z)}.$$

• Suppose that $\alpha = 1$, by Lemma 4.6, there exists a sequence exists a sequence $(f_m)_m$ in G such that $\underset{m\longrightarrow +\infty}{lim} f_m = f = T_a \in G_1$. So $T_a(z) = z + a \in \overline{G(z)}$. It follows that $\alpha z + a \in \overline{G(z)}$ and so $z + E_G \subset \overline{G(z)}$. This proves that $\overline{\Lambda_G}.z + E_G \subset \overline{G(z)}$.

Conversely, let's prove that $G(z) \subset \Lambda_G.z + E_G$. Let $f \in G$.
• Suppose that $f = (a, \lambda) \in G\backslash\mathcal{T}_n$. By Lemma 2.6.(i), $f(0) = (1-\lambda)a \in E_G$ since E_G is a vector space. Then $f(z) = \lambda(z-a) + a = \lambda z + (1-\lambda)a \in \Lambda_G.z + E_G$.
• Suppose that $f = T_a \in G \cap \mathcal{T}_n$, so $f(z) = z + a \in \Lambda_G.z + E_G$, since by Lemma 2.6.(i), $f(0) = a \in E_G$. It follows that $G(z) \subset \Lambda_G.z + E_G$. Therefore $\overline{G(z)} \subset \overline{\Lambda_G}.z + E_G$. Hence $\overline{G(z)} = \overline{\Lambda_G}.z + E_G$. □

5. **Some results in the case** $G \subset \mathcal{SR}_n$

In this section G is a subgroup of $\mathcal{S}_i\mathcal{R}_n$ ($i = 2$ or $i = 3$).

Lemma 5.1. *Let G be a subgroup of $\mathcal{H}(n, \mathbb{C})$ such that $G \subset \mathcal{S}_i\mathcal{R}_n$ ($i = 2$ or $i = 3$). Then:*
(i) $\Lambda_G = F_i$. Moreover, for every $\lambda, \mu \in \Lambda_G$, $\lambda = \mu^k$ for some $k \in \mathbb{Z}$.
(ii) $G_1(0) = G(0)$.
(iii) There exists $a \in \Gamma_G$, such that $G(z) = \Lambda_G(z-a) + G(a)$, for every $z \in \mathbb{C}^n$.

Proof. Let $a \in \Gamma_G$ and $G' = T_{-a} \circ G \circ T_a$. Then $h = \mu id_{\mathbb{C}^n} \in G'$, for some $\mu \in \Lambda_G$, so $0 \in \Gamma_G$.
(i) The proof follows from the construction of $\mathcal{S}_i\mathcal{R}_n$, $i = 2$ or 3 and since F_i is cyclic.
(ii) Firstly, $\Gamma_{G'} \subset G'_1(0)$; Indeed, if $f = (b, \lambda) \in G'\backslash\mathcal{T}_n$, then by (i), $\lambda^k = 1$ for some $k \in \mathbb{Z}$ since F_i is cyclic, $i \in \{2, 3\}$. Thus $f^k = (b, 1) = T_b$ and so $b \in G'_1(0)$.
Secondly, let $a \in G'(0)\backslash G'_1(0)$ and $f = (b, \lambda) \in G'\backslash\mathcal{T}_n$ such that $a = f(0) = (1-\lambda)b$. By (i), $\mu^k = \lambda$, for some $k \in \mathbb{Z}$. By applying Lemma 2.10.(iv) on the group G_k generated by h^k and f, we have $(1-\lambda)\Gamma_{G^k} \subset \Gamma_{G^k}$, so $a = (1-\lambda)b \in \Gamma_{G^k} \subset \Gamma_{G'}$. It follows that $a \in \Gamma_{G'} \subset G'_1(0)$. The proof of (ii) is complete.
(iii) By Lemma 2.11,(i), $G'(z) \subset \Lambda_G z + G'(0)$. Conversely, let $\lambda \in \Lambda_{G'}$, $a \in G'_1(0)$ so $T_a \in G'_1$. By (i), $\lambda = \mu^k$ for some $k \in \mathbb{Z}$. As in the proof of Lemma 2.10.(iv), $g = T_a \circ h^k = \left(\frac{a}{1-\lambda},\ \lambda\right)$, so $a' = \frac{a}{1-\lambda} \in \Gamma_{G'}$. Therefore, $g(z) = \lambda(z-a') + a' = \lambda z + (1-\lambda)a'$, so $g(z) = \lambda z + a \in G'(z)$. Therefore, $\Lambda_{G'}z + G'_1(0) \subset G'(z)$. By (ii), $\Lambda_{G'}z + G'(0) \subset G'(z)$. It follows that $G'(z) = \Lambda_{G'}z + G'(0)$, then $G(z) = \Lambda_G(z-a) + G(0)$, since $\Lambda_{G'} = \Lambda_G$, $G(z) = T_a(G'(z-a))$ and $G(a) = T_a(G'(0))$. The proof of (iii) is complete. □

6. **Proof of main results**

Recall that $U = \mathbb{C}^n \backslash E_G$.

Proof of Theorem 1.1. Let $a \in E_G$ and $G' = T_{-a} \circ G \circ T_a$. By Lemma 2.5.(iii), $E_{G'} = T_{-a}(E_G)$ is a vector subspace of $\mathbb{C}^n$. Then :

• The Proof of (1).(i) results from Proposition 4.2.

• Proof of (1).(ii): By Proposition 4.5, $\overline{G'(z-a)} = \overline{\Lambda_{G'}}.(z-a) + E'_G$, for every $z \in U$. So by Lemma 2.5,(ii), $T_{-a}(\overline{G(z)}) = \overline{\Lambda_G}.(z-a) + E_G - a$, it follows that $\overline{G(z)} = \overline{\Lambda_G}.(z-a) + E_G$.

• *Proof of* (2)*:* The proof of (2) results from Lemma 5.1. □

We will use the following Lemmas to prove Corollary 1.2.

Lemma 6.1. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{C})$ with $G \backslash \mathcal{R}_n \neq \emptyset$ and $U \neq \emptyset$, then for every $z \in \overline{G(y)} \cap U$ we have $\overline{G(z)} \cap U = \overline{G(y)} \cap U$.*

Proof. Suppose that E_G is a vector space (otherwise, by Lemma 2.5.(ii), we can replace G by $G' = T_{-a} \circ G \circ T_a$ for some $a \in E_G$). Let $z \in \overline{G(y)} \cap U$ and $y \in \overline{G(z)} \cap U$. By Theorem 1.1.(1).(iii), there exists $a \in E_G$ such that $\overline{G(z)} = \overline{\Lambda_G}(z-a) + E_G$. Since E_G is a vector space and $a \in E_G$ then $\overline{G(z)} = \overline{\Lambda_G} z + E_G$. In the same way,

$$\overline{G(y)} = \overline{\Lambda_G} y + E_G, \qquad (1).$$

See that $\overline{G(z)} \cap U = (\overline{\Lambda_G} \backslash \{0\}) z + E_G$. Write $y = \alpha z + b$, where $\alpha \in \overline{\Lambda_G} \backslash \{0\}$ and $b \in E_G$. So by (1),

$$\overline{G(y)} = \overline{\Lambda_G} y + E_G = \overline{\Lambda_G}(\alpha z + b) + E_G = \alpha \overline{\Lambda_G} z + E_G.$$

By Lemma 2.4, $0 \in \Lambda_G$ and Λ_G is a subgroup of $\mathbb{C}^*$, then $\alpha \overline{\Lambda_G} = \overline{\Lambda_G}$, since $\alpha \in \Lambda_G$. Therefore $\overline{G(y)} = \overline{\Lambda_G} z + E_G = \overline{G(z)}$. □

Lemma 6.2. *Let G be a non abelian subgroup of $\mathcal{H}(n, \mathbb{C})$ such that E_G is a vector subspace of $\mathbb{C}^n$. Let $z \in U$ then the vector subspace $H_z = \mathbb{C} z \oplus E_G$ of $\mathbb{C}^n$ is G-invariant.*

Proof. Let $z \in \mathbb{C}^n \backslash E_G$ and $f \in G$ having the form $f(z) = \lambda z + a$, $z \in \mathbb{C}^n$, one has $a = f(0) \in E_G$. For every $\alpha \in \mathbb{C}$, $b \in E_G$, we have $f(\alpha z + b) = \lambda(\alpha z + b) + a = \lambda \alpha z + \lambda b + a$. Since E_G is a vector space, then $\lambda b + a \in E_G$ and so $f(\alpha z + b) \in H_z$. □

Proof of Corollary 1.2.

• *The proof of* (1).(i)*:* The proof results from Lemma 6.1.

• *The proof of* (1).(ii)*:* As $G \backslash \mathcal{R}_n \neq \emptyset$, then by Lemma 2.4, $0 \in \overline{\Lambda_G}$. So the proof of (ii) results from Theorem 1,1.(1).(ii).

• *The proof of* (1).(iii)*:* Suppose that E_G is a vector subspace of $\mathbb{C}^n$ (leaving, by Lemma 2.5, to replace G by $G' = T_{-a} \circ G \circ T_a$, for some $a \in E_G$).

Recall that $U = \mathbb{C}^n \backslash E_G$ and let $z, y \in U$ with $z \neq y$. Denote by $H_z = \mathbb{C}.z \oplus E_G$ and by $H_y = \mathbb{C}.y \oplus E_G$. By lemma 6.2 we have H_z and H_y are G-invariant. Let $\Phi : \ H_z \longrightarrow H_y$ be the homeomorphism defined by $\Phi(\alpha z + v) = \alpha y + v$ for every $\Phi \in \mathbb{C}$ and $v \in E_G$. For every $f \in G$, with the form $f(z) = \lambda z + a$, $z \in \mathbb{C}^n$, then

by Lemma 3.9.(i), $a = f(0) \in E_G$ and so $\Phi(f(z)) = \Phi(\lambda z + a) = \lambda y + a = f(y)$. It follows that $\Phi(G(z)) = G(y)$.
• *The proof of* (2)*:* The proof of (2) results from Lemma 5.1. $\square$

Proof of Corollary 1.3.
• From Corollary 1.2.(ii), the closure of every orbit of G contains E_G. Since $\dim(E_G) \geq 1$, G has no discrete orbit. $\square$

Proof of Corollary 1.4. The proof of Corollary 1.4 results from Theorem 1.1 and Corollary 1.2 and the fact that $\overline{U} = \mathbb{C}^n$ if $U \neq \emptyset$. $\square$

Proof of Corollary 1.7. If G is generated by $f_1 = (a_1, \lambda_1), \dots, f_{n-2} = (a_{n-2}, \lambda_{n-2}) \in \mathcal{H}(n, \mathbb{C})$. By Lemma 2.2, $E_G \subset Vect(a_1, \dots, a_{n-2})$, so $\dim(E_G) \leq n-2$. By Theorem 1.1 there are two cases:
• If $G \backslash \mathcal{SR}_n \neq \emptyset$, then $G(z) = \Lambda_G z + E_G \subset \mathbb{C}z + E_G$, for every $z \in \mathbb{C}^n \backslash E_G$ and $\overline{G(z)} = E_G$ for every $z \in E_G$. Therefore, $\overline{G(z)} \neq \mathbb{C}^n$.
• If $G \subset \mathcal{SR}_n$, then $\overline{G(z)} = \mathbb{C}^n$, for some $z \in \mathbb{C}^n$ if and only if $\overline{G_1(0)} = \mathbb{C}^n$. Since $G_1(0) \subset E_G$, it follows that G has no dense orbit.

7. Examples

Example 7.1. Let G be the non abelian subgroup of $\mathcal{H}(1, \mathbb{C})$ generated by T_a the translation by $a \in \mathbb{C}^*$ and $h = e^{i\theta} I_2$, $\theta \notin \pi\mathbb{Z}$. Then:
(i) If $\theta \in H_2 \cup H_3$ then every orbit of G is closed and discrete.
(ii) If $\theta \notin H_2 \cup H_3$ then every orbit of G is dense in $\mathbb{C}$.

Proof. Firstly, remark that G is generated by h and $g = T_a \circ f = \left(\frac{a}{1-e^{i\theta}},\ e^{i\theta}\right)$.
- If $\theta \in H_2 \cup H_3$, then $g \in \mathcal{SR}_1$, by Theorem 3.2.(ii), the property (i) follows.
- If $\theta \notin H_2 \cup H_3$, then $g \notin \mathcal{SR}_1$, by Theorem 3.2.(i), the property (ii) follows.
$\square$

Example 7.2. Let G be a subgroup of $\mathcal{H}(2, \mathbb{C})$ generated by $f_1 = (a_1, \alpha_1)$ and $f_2 = (a_2, \alpha_2)$ and $f_3 = (a_3, \alpha_3)$, where $\alpha_k \in \mathbb{C} \backslash \mathbb{R}$ with $|\alpha_k| \neq 1$, for every $1 \leq k \leq 3$ and $a_1 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $a_3 = \begin{bmatrix} -\sqrt{3} \\ -\sqrt{2} \end{bmatrix}$. Then every orbit of G is dense in $\mathbb{C}$.

Indeed, by Lemma 2.1.(i), G is non abelian. By Proposition 4.2, for every $z \in E_G$, we have $\overline{G(z)} = E_G$. By Remark 2.3 $E_G = \mathbb{C}^2$, so by Theorem 1.1, every orbit of G is dense in $\mathbb{C}^2$.

Example 7.3. Let $(a_1, \dots, a_n)$ be a basis of $\mathbb{C}^n$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Then every orbit of the group generated by $T_{a_1}, \dots, T_{a_n}$, λId is dense in $\mathbb{C}^n$.

Indeed, by Remark 2.3 we have $E_G = \mathbb{C}^n$ and by Proposition 4.2 every orbit of G is dense in $\mathbb{C}^n$.

References

1. N.I.Zhukova, *Minimal sets of Cartan Foliations*, Trudy Mathematicheskogo Institua imeni V.A. Steklova (2007), vol 256, 115147.
2. Arek Goetz, *Dynamics of piecwise rotations*, Continuous ad Discrete Dynamical Systems, (1998),4(4), 593-608.
3. Michael Boshernitzan and Isaac Kornfeld. *Interval translation mappings*, Ergodic Theory and Dynamical Systems, (1995), 15,821-831.
4. Waldschmidt.M, Topologie des points rationnels., Cours de troisième Cycle, Université P. et M. Curie (Paris VI), 1994/95.
5. A.Ayadi and Y.N'dao, *Non abelian group generated by affine homotheties acting on* $\mathbb{R}^n$, preprint, arXiv 2010.

Yahya N'dao, University of Moncton, Faculty of sciences, Moncton, Canada

Adlene Ayadi, University of Gafsa, Faculty of sciences, Department of Mathematics,Gafsa, Tunisia.
E-mail address: yahiandao@yahoo.fr , adleneso@yahoo.fr